Part 2.2 Continuous functions and their properties

v1 2019-20

Intermediate Values

Theorem 2.2.1 *(Bolzano 1817)* ***Intermediate Value Theorem***
Suppose that f is a function continuous on a closed and bounded interval $[a, b]$.

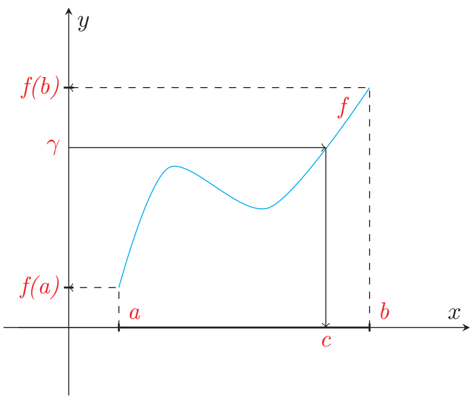
For all γ between $f(a)$ and $f(b)$ there exist $c : a \le c \le b$ for which $f(c) = \gamma$.

Here 'between' means $f(a) \le \gamma \le f(b)$ if $f(a) \le f(b)$, $f(b) \le \gamma \le f(a)$ otherwise.

Important Do get the order of the quantifiers correct, "for all" first and "there exists" second, i.e.

$$\forall \gamma \text{ between } f(a) \text{ and } f(b)\ \exists\, c : a \le c \le b \text{ and } f(c) = \gamma.$$

On a graph you would be starting with a point γ on the y-axis and finding a point c on the x-axis which maps to it.



Before the proof recall $\mathbb{R}$ is *complete*. This means that every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound. That is:

$$(A \subseteq \mathbb{R} : A \ne \phi \text{ and } \exists\, M : \forall a \in A, a \le M) \implies \text{lub}A \text{ exists.}$$

And the definition of lub$A$ is that, if $\lambda = \text{lub}A$ then

- λ is an *upper bound*: $\forall a \in A, a \le \lambda$,
- λ is the *least* of all upper bounds; if μ is an upper bound for A then $\lambda \le \mu$.

Alternatively

- For all $\delta > 0$, $\lambda - \delta$ is **not** an upper bound for A which means $\exists\, a \in A : \lambda - \delta < a \le \lambda$.

Proof of I V Th$^{\text{m}}$ We first 'translate and reflect' the function f. There are two cases;

- If $f(a) \le f(b)$ then $f(a) \le \gamma \le f(b)$. Define $g(x) = f(x) - \gamma$, then $g(a) \le 0$ and $g(b) \ge 0$.
- If $f(a) > f(b)$ then $f(b) \le \gamma < f(a)$. This time define $g(x) = \gamma - f(x)$, then again $g(a) \le 0$ and $g(b) \ge 0$.

  Summing up, define

$$g(x) = \begin{cases} f(x) - \gamma & \text{if } f(a) \le f(b) \\ \gamma - f(x) & \text{if } f(a) > f(b)\,. \end{cases}$$

Then $g(a) \le 0 \le g(b)$.

If either $g(a) = 0$ or $g(b) = 0$ the proof is finished, simply choose $c = a$ or b respectively.

Thus we may assume that we have *strict* inequalities in $g(a) < 0 < g(b)$ and it suffices to find $c \in (a,b) : g(c) = 0$.

Consider the set

$$\mathcal{S} = \{x \in [a,b] : g(x) < 0\}\,.$$

Then $\mathcal{S} \ne \phi$ since $a \in \mathcal{S}$, while $\mathcal{S} \subseteq [a,b]$ and so $\mathcal{S}$ is bounded above by b. Therefore, by the Completeness Axiom of $\mathbb{R}$, there exists $c \in \mathbb{R} : c = \operatorname{lub} \mathcal{S}$.

We want to first show that $c \in (a,b)$, i.e. $c \ne a$ or b. From the definition of a function being continuous on a *closed* interval we have $\lim_{x\to a+} g(x) = g(a)$ and $\lim_{x\to b-} g(x) = g(b)$.

Following the method of an earlier lemma we choose $\varepsilon = |g(a)|/2 > 0$ in the definition of. $\lim_{x\to a+} g(x) = gv\,(a)$, to find $\delta_1 > 0$ such that if $a < x < a + \delta_1$ then $g(x) < g(a)/2 < 0$. This means $[a, a+\delta_1) \subseteq \mathcal{S}$ and so $c \ge a + \delta_1$.

Similarly, choosing $\varepsilon = g(b)/2 > 0$ in the definition of. $\lim_{x\to b-} g(x) = g(b)$, we find $\delta_2 > 0$ such that if $b - \delta_2 < x \le b$ then $g(x) > g(b)/2 > 0$. this means all such $x \notin \mathcal{S}$ and so $c \le b - \delta_2$.

From these two observations we deduce that $c \in (a, b)$.

Let $\varepsilon > 0$ be given. Since g is continuous at c there exists $\delta > 0$ such that if $|x - c| < \delta$ then $|g(x) - g(c)| < \varepsilon$. That is

$$c - \delta < x < c + \delta \implies g(x) - \varepsilon < g(c) < g(x) + \varepsilon. \tag{1}$$

First, choose $x_1 = c + \delta/2$. Then (1) implies $g(c) > g(x_1) - \varepsilon$. Yet $x_1 > c$, an **upper bound** on $\mathcal{S}$ and so $x_1 \notin \mathcal{S}$, that is $g(x_1) \geq 0$. Combine to get $g(c) > -\varepsilon$.

Next, since $c - \delta < c$, the **least** upper bound on $\mathcal{S}$, we have that $c - \delta$ is not an upper bound on $\mathcal{S}$, i.e. there exists some $x_2 \in \mathcal{S}$ satisfying $c - \delta < x_2 < c$. Then (1) implies $g(c) < g(x_2) + \varepsilon$. Yet $x_2 \in \mathcal{S}$ implies $g(x_2) < 0$. Combine as $g(c) < \varepsilon$.

Further combine to get $-\varepsilon < g(c) < \varepsilon$. True for all $\varepsilon > 0$ implies $g(c) = 0$. ■

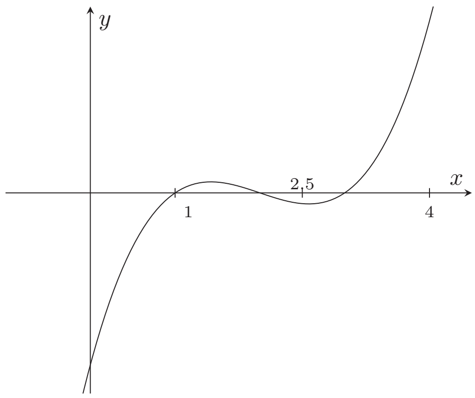
There is a good chance you will have used this result, for example by finding roots of a polynomial by looking for a sign change.

Example 2.2.2 *Let $p(x) = x^3 - 6x^2 + 11x - 6$. Show that there is a zero of this polynomial between 0 and 4. Is there a zero between 0 and 2.5?*

Solution $p(0) = -6$ and $p(4) = 6$ so $p(0) < 0 < p(4)$, i.e. 0 is an intermediate value between $p(0)$ and $p(4)$. Since p is a polynomial it is continuous so we can apply the Intermediate Value Theorem with $\gamma = 0$ to deduce that there exists $0 < c < 4$ for which $p(c) = 0$.

Since $p(2.5) = -0.375$ there is no sign change between 0 and 2.5 so we cannot apply the Intermediate Value Theorem with $\gamma = 0$ to show there is a zero in $[0, 2.5]$. This is a weakness of this method to find roots for it is not hard to see that $x = 1$ is a root of $p(x)$ in $[0, 2.5]$. ■

In fact, from the graph you can see two roots between 0 and 2.5.



Example 2.2.3 *Show that for all real* $a, b > 0$ *there is a solution to* $a \sin x = b \cos x$ *in* $[0, \pi/2]$.

Solution in Tutorial Let $f(x) = a \sin x - b \cos x$. We see that $f(0) = -b$ and $f(\pi/2) = a$ so $f(0) < 0 < f(\pi/2)$. Since f is continuous on $[0, \pi/2]$ the Intermediate Value Theorem implies there exists $c \in (0, \pi/2)$ such that $f(c) = 0$, i.e. $a \sin c = b \cos c$. ■

Example 2.2.4 *(A special case of the)* ***Fixed Point Theorem****. If* $f : [0, 1] \to [0, 1]$ *is continuous then there exists* $c \in [0, 1]$ *such that* $f(c) = c$.

Solution Define $g(x) = f(x) - x$, a function continuous on $[0, 1]$. By definition $0 \le f(x) \le 1$ for all $0 \le x \le 1$. In particular $f(0) \ge 0$ and so

$$g(0) = f(0) - 0 \ge 0.$$

Similarly, $f(1) \le 1$ so

$$g(1) = f(1) - 1 \le 1 - 1 = 0.$$

That is, $g(1) \le 0 \le g(0)$. So apply I.V.Thm to g on $[0, 1]$ to find $c : g(c) = 0$, i.e. $f(c) = c$. ■

This result should not be a surprise. Being continuous on a closed interval the function f is 'tied down' at $f(0)$ and $f(1)$. Since these values are between 0 and 1 the graph between them has to cross the line $y = x$. See Figure 1.

The same result should hold with $y = x$ replaced by any continuous function between $(0,0)$ and $(1,1)$. For example see Figure 2 where $y = x^3$.

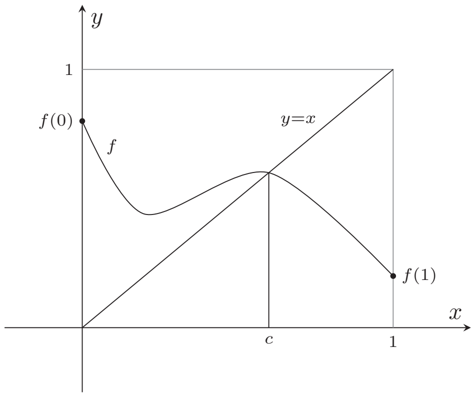


Figure 1: $y = x$

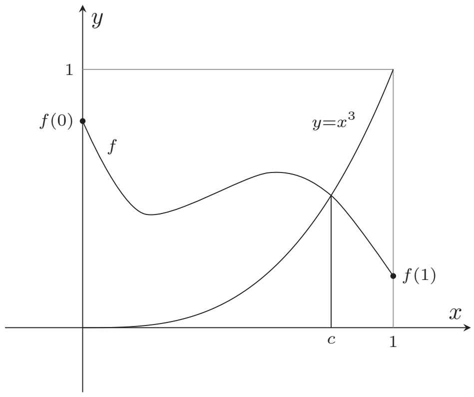
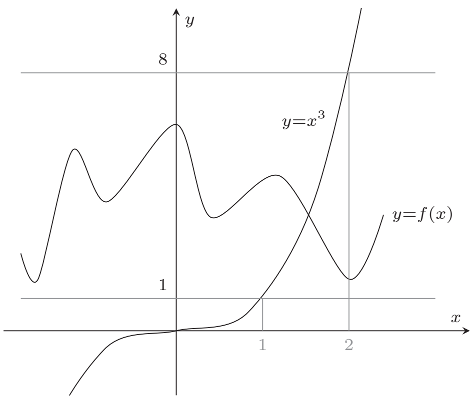


Figure 2: $y = x^3$

Example 2.2.5 *If $f : \mathbb{R} \to [1,8]$ is continuous then there exists $c \in \mathbb{R}$ such that $f(c) = c^3$.*

Solution in Tutorial If there is a solution of $f(c) = c^3$ then, since $1 \leq f(c) \leq 8$ we have $1 \leq c^3 \leq 8$, i.e. $1 \leq c \leq 2$. This could be seen on the graph:



So we need only apply the Intermediate Value Theorem on the interval $[1,2]$.

Let $g(x) = f(x) - x^3$.

Then $g(1) = f(1) - 1^3 \geq 1 - 1 = 0$ since $f(x) \geq 1$ for all $x \in [1,2]$.

Also $g(2) = f(2) - 2^3 \leq 8 - 8 = 0$ since $f(x) \geq 8$ for all $x \in [1,2]$.

Thus $g(1) \geq 0 \geq g(2)$, i.e. 0 is an intermediate value. Apply the Intermediate Value Theorem to g on $[1, 2]$ with $\gamma = 0$ to show there exists $c \in [1, 2]$ such that $g(c) = 0$, that is, $f(c) = c^3$. ■

Bounded Functions

Definition 2.2.6 *A function f is said to be* ***bounded on the interval*** $[a, b]$ *if there exist numbers L and U such that $L \leq f(x) \leq U$ for all $a \leq x \leq b$. That is*

$$\exists\, L, U \in \mathbb{R} : \forall x \in [a, b]\,, L \leq f(x) \leq U.$$

Alternatively, there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $a \leq x \leq b$ i.e.

$$\exists\, M \in \mathbb{R} : \forall x \in [a, b]\,, |f(x)| \leq M.$$

A function f is said to ***attain its lower bound*** *on the interval $[a, b]$ if there exists $c \in [a, b]$ such that $f(c) \leq f(x)$ for all $a \leq x \leq b$, i.e*

$$\exists\, c \in [a, b] : \forall x \in [a, b]\,, f(c) \leq f(x)\,.$$

A function f is said to ***attain its upper bound*** *on the interval $[a, b]$ if there exists $d \in [a, b]$ such that $f(x) \leq f(d)$ for all $a \leq x \leq b$, i.e*

$$\exists\, d \in [a, b] : \forall x \in [a, b]\,, f(d) \geq f(x)\,.$$

Recall that we previously stated, without proof, that

• $\lim_{x\to a} f(x) = L$ if, and only if, $f(y_n) \to L$ as $n \to \infty$ for all sequences $\{y_n\}_{n\geq1}$ with $y_n \neq a$ for all $n \geq 1$ and $y_n \to a$ as $n \to \infty$.

Because f is continuous at a if, and only if, $\lim_{x\to a} f(x) = f(a)$ we get

• f **is continuous** at a iff $f(y_n) \to f(a)$ as $n \to \infty$ for all sequences $\{y_n\}_{n\geq1}$ with $y_n \to a$ as $n \to \infty$.

(There is no need to exclude $y_n = a$ since f is defined at a.)

We will make use of sequences to prove a boundedness result but first we need an important result from the theory of sequences.

Definition 2.2.7 *Given a sequence a* ***subsequence*** *remains after deleting elements from the sequence.*

Thus given a sequence $\{x_n\}_{n\geq1}$ a subsequence is denoted by $\{x_{n_k}\}_{k\geq1}$, where $1 \leq n_1 < n_2 < n_3 < ....$ So n_k is the k-th term remaining after some terms have been removed from the original sequence. If none of the first k terms are removed than $n_k = k$. If any of the first k terms had been removed than $n_k > k$. Hence $n_k \geq k$ for all $k \geq 1$.

Theorem 2.2.8 ***The Bolzano-Weierstrass Theorem (1817)*** *A bounded sequence of real numbers has a convergent subsequence.*

Proof not given in lectures. It was also stated without proof in MATH10242. But see the Appendix. ■

Theorem 2.2.9 *A function continuous on a closed, bounded interval,* $[a, b]$ *, is bounded.*

Proof by contradiction. The definition of bounded on an interval is

$$\exists M \geq 0, \forall x : a \leq x \leq b \implies |f(x)| \leq M.$$

The negation of this is

$$\forall M \geq 0, \exists x : a \leq x \leq b \text{ and } |f(x)| > M. \quad (2)$$

(Recall from truth tables that we have the logical equivalence

$$\text{not}\,(p \Rightarrow q) \equiv p \text{ and } (\text{not } q)$$

for propositions p and q.)

We assume (2) for contradiction and apply it repeatedly with $M = n \in \mathbb{N}$, to find points $x_n : a \leq x_n \leq b$ and $|f(x_n)| > n$.

We thus get a sequence $\{x_n\}_{n\geq 1}$.

The points of this sequence satisfy $a \leq x_n \leq b$, and so it is a bounded sequence. Thus by the Bolzano-Weierstrass Theorem it has a *convergent* subsequence $\{x_{n_k}\}_{k\geq 1}$. Let c be the limit of this sequence, i.e.

$$c = \lim_{k\to\infty} x_{n_k}.$$

Then $a \leq c \leq b$ since $a \leq x_{n_k} \leq b$ for all $k \geq 1$. Since f is continuous on $[a, b]$ we have, as noted above, that

$$\lim_{k\to\infty} f(x_{n_k}) = f(c)\,. \quad (3)$$

But, by definition of the sequence, we have

$$|f(x_{n_k})| > n_k, \quad (4)$$

while $n_k \geq k$ for all k implies that $n_k \to \infty$ as $k \to \infty$. So (4) tells us that $\{f(x_{n_k})\}_k$ is an unbounded sequence, i.e. it diverges, while (3) tells us converges to a finite value, $f(c)$. This contradiction means our assumption is false and thus f is bounded. ■

Can we remove any of the assumptions in the Theorem and still deduce that f is bounded?

Example 2.2.10 *$f(x) = 1/x$ on $(0, 1]$ is continuous but not bounded.*

So it is important in Theorem 2 that the interval $[a, b]$ is closed.

Example 2.2.11 *$f(x) = x$ on $[1, \infty)$ is continuous but not bounded.*

So it is important in Theorem 2 that the interval $[a, b]$ is bounded.

To sum up, f continuous on a

closed and bounded interval $\Longrightarrow$ f is bounded,
closed interval $\not\Longrightarrow$ f is bounded,
bounded interval $\not\Longrightarrow$ f is bounded.

Given that a continuous function on a closed interval is bounded the proof we give that it *attains* its bounds depends on a **TRICK**.

Theorem 2.2.12 *Suppose that f is a function continuous on a closed and bounded interval $[a, b]$. Then there exist $c, d \in [a, b]$ such that*

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$.

So the upper and lower bounds for f are *attained* at $x = d$ and $x = c$ and we can talk about the maximum and minimum values of f.

Proof Since f is a function continuous on a closed interval $[a, b]$ it is bounded by the previous Theorem, and thus the set of real numbers $\{f(x) : a \leq x \leq b\}$ is bounded. Since this set is non-empty the Completeness axiom implies that the set has a least upper bound. Let

$$M = \text{lub}\,\{f(x) : a \leq x \leq b\},$$

so $f(x) \leq M$ for all $a \leq x \leq b$.

Assume for a contradiction that M is **not** attained, i.e. $f(x) < M$ for all $a \leq x \leq b$. Then $M - f(x) > 0$ in which case

$$g(x) := \frac{1}{M - f(x)}$$

is well-defined on $[a,b]$. By the rules for continuous functions g is continuous on $[a,b]$. Hence, by the previous Theorem, g is bounded above. That is, there exists $K>0$ say, such that

$$\frac{1}{M-f(x)} \leq K$$

for all $a \leq x \leq b$. This rearranges to give

$$f(x) \leq M - \frac{1}{K},$$

for all $a \leq x \leq b$, i.e. $M - 1/K$ is *an* upper bound for $\{f(x) : a \leq x \leq b\}$. But this contradicts the fact that M is the *least* of all upper bounds for this set. Thus our assumption is false, i.e. M **is** attained. That is, there exists $d \in [a,b]$ such that

$$f(d) = M \geq f(x)$$

for all $x \in [a,b]$, since M is an upper bound for f on $[a,b]$.

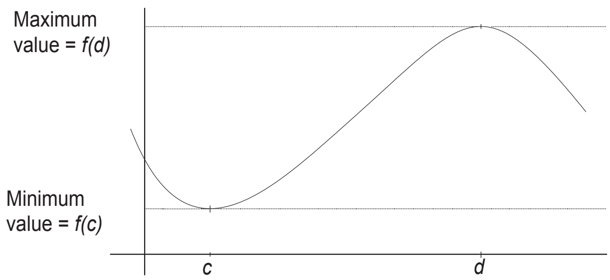
I leave it to the student (and the tutorial) to show that the greatest lower bound of f on $[a,b]$ is attained.

■

Combining the last two results and we have

Theorem 2.2.13 ***Boundedness Theorem*** *(1861) A function continuous on a closed, bounded interval,* $[a,b]$, *is bounded and attains its bounds.* ■

In fact, if f is continuous on a closed interval $[a,b]$ then f takes on *every* value between the maximum and minimum values of f, a result not proved in this course. In other words the image set $f([a,b])$ is a closed interval $[f(e), f(k)]$ or, more succinctly, "*the continuous image of a closed interval is a closed interval*".



Note It is important for these last two results that we have f is continuous, the domain, $[a,b]$, is closed, and the domain, $[a,b]$, is bounded. If any of these three conditions fail to hold the conclusion may well not hold.